

Group Theory in the Bedroom

by Marcio Luis Ferreira Nascimento

How can we flip a mattress, so as to keep it durable, and how many ways can it be done?

This question is a great way to introduce the intuitive flipping laws of a mattress, as shown by the American writer and computer scientist Brian Patrick Hayes in his incredible book *Group Theory in the Bedroom, and Other Mathematical Diversions* (2009). The answer, which takes into account the initial condition of the mattress, involves a mathematical theory established by an adolescent who died in his twenties and covers applications as relevant and disparate as mattress flipping and quantum physics.

Rest is always necessary, and sleeping comfortably can be considered as a pleasure. So choosing a mattress is therefore one of most important tasks when buying a new bed.

My wife and I looked at various price options, types and manufacturers. An assistant at a particular store even claimed that it was not necessary to flip the mattress many times because it was very resistant. But if it were necessary to flip the mattress a few times during the year, how many flips are necessary or available? As another customer explained to us, there are in fact only four situations to consider!

Briefly, the explanation was the following: first there is the initial state. Then the mattress can be rotated around any of three orthogonal axes. So the second state arises from rotating the mattress by 180° around a vertical axis, which simply means changing the sleeping position between head and feet as the mattress is still the same way up. The third state corresponds to a rotation of 180° around the shorter horizontal mattress axis (imagining that it is rectangular), which turns the mattress over so that the headboard position of the mattress is now at the bottom of the bed. Finally, the fourth and last way would be a rotation of 180° around the longer horizontal mattress axis, again turning the mattress over and changing the right side to the opposite. Figure 1 shows all these four mathematical operations.

The first of these mathematical operations consists of a rotation that does not imply change, labelled as I , where the initial and final positions are the same. The p -turn combines the letters a, b (headboard end of the bed), c, d (bottom of the bed) that identify the corners of the mattress so as to change them in position by reversing

Action	Initial position	Final position
I	$\begin{array}{cc} a & b \\ c & d \end{array}$	$\begin{array}{cc} a & b \\ c & d \end{array}$
p	$\begin{array}{cc} a & b \\ c & d \end{array}$	$\begin{array}{cc} c & d \\ a & b \end{array}$
q	$\begin{array}{cc} a & b \\ c & d \end{array}$	$\begin{array}{cc} b & a \\ d & c \end{array}$
r	$\begin{array}{cc} a & b \\ c & d \end{array}$	$\begin{array}{cc} d & c \\ b & a \end{array}$

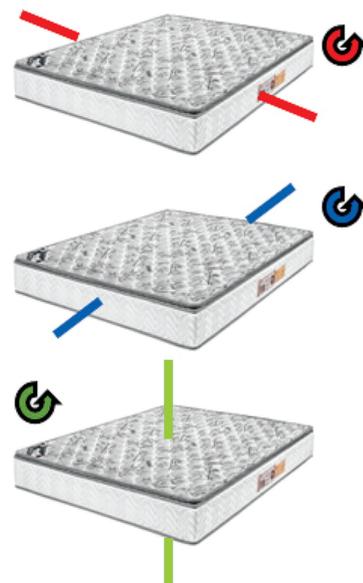


Fig. 1 All flips (actions or rotations) for a rectangular mattress

the letters, matching so that a and c change places, as well as b and d . The q -turn reverses the letters and combines them so that a and b change places, as well as c and d . The r rotation only inverts the letters, changing a by d and b by c . Some curious results arise when two operations are combined in sequence:

- if the same operation is repeated, i.e. p followed by p , q followed by q or r followed by r , all result in the

initial condition I – and this should not be surprising, as turning the mattress and repeating the same operation returns it to the initial state;

- the order of two different operations in sequence is irrelevant, since q followed by p results in the same as p followed by q , and in fact, the result is r ;
- combining p followed by r results in q ;
- combining r followed by q results in p .

Note that if one turns the mattress back 180° in any of the last three situations, it returns to the original condition. This requirement is mathematically defined as *closure*, which says the set of operations is in some sense complete. It's another important reminder: any other type of flipping of the mattress will invariably result in one of these four ways, even when they are combined in sequence! And more: any combination of two basic operations can be replaced by a single operation.

As incredible as it may seem, this apparent story actually corresponds to a new, elegant and sophisticated mathematics created by the brilliant, romantic and naive French teenager mathematician Évariste Galois (1811–1832), see Figure 2, who unfortunately died in a duel at the age of 20.



Fig. 2 Galois as sketched posthumously by his brother Alfred, published in *Le Magasin Pittoresque* 16 (1848) pp. 227–8
[Figure in public domain]

Galois elaborated in a unique and original way, a new branch of mathematics now called *Group Theory*, which is also known as the mathematics of symmetries, and has numerous applications. Most of the works of Galois were published posthumously by the French mathematician Joseph Liouville (1809–1882) (Liouville, 1846; Galois, 1846). The reader can learn a lot of group symmetry by means of the excellent books by Ian Stewart (Stewart, 2015) and the Israeli-American astrophysicist Mario Livio (Livio, 2006).

In the world of very small things, as in the field of quantum physics, there is a rule of nature in which the commutative law ceases to exist between two operators representing a pair of variables. The effect is that these

variables cannot be both known and measured at the same time. This is called the 'Uncertainty Principle', and it was established in 1927 by the German physicist Werner Karl Heisenberg (1901–1976). Heisenberg went on to win the 1932 Nobel Prize for Physics.

In summary, mathematics exists to help and better understand the world, considering the very small particles of the universe, or even an ordinary mattress.

It is interesting to note that the mathematical rotations we have discussed are in accordance with the standards of some good mattress manufacturers, i.e. it is important to rotate them every three months, which of course means that each of the four positions is visited once a year. And incidentally, in the end we did buy a good mattress, which is providing a good night's rest.

References

- Galois, E. 1846 'Articles Publiés par Galois dans les Annales de Mathématiques de M. Gergonne: Demonstration d'un Theorème sur les Fractions Continues Périodiques', *J. Math. Pures App.* **11**, pp. 385–394 (in French).
- Galois, E. 1846 'Articles Publiés par Galois dans les Bulletin des Sciences Mathématiques de M. Ferussac: Analyse d'un Mémoire sur la Résolution Algébrique des Équations', *J. Math. Pures App.* **11**, pp. 395–407 (in French).
- Galois, E. 1846 'Lettre de Galois à M. Auguste Chevalier', *J. Math. Pures App.* **11**, pp. 408–416 (in French).
- Galois, E. 1846 'Mémoire Sur les Conditions de Résolubilité des Équations par Radicaux', *J. Math. Pures App.* **11**, pp. 417–433 (in French).
- Galois, E. 1846 'Fragment d'un Second Mémoire: Des Équations Primitives qui Sont Solubles par Radicaux', *J. Math. Pures App.* **11**, pp. 434–444 (in French).
- Hayes, B. 2009 *Group Theory in the Bedroom, and Other Mathematical Diversions*, Hill and Wang, New York, 288 pages.
- Heisenberg, W. 1927 'Über den Anschaulichen Inhalt der Quantentheoretischen Kinematik und Mechanik (Perceptual Content of Quantum Theoretical Kinematics and Mechanics)', *Zeit. Phys.* **43**, pp. 172–198 (in German).
- Liouville, J. 1846 'Oeuvres Mathématiques d'Évariste Galois', *J. Math. Pures App.* **11**, pp. 381–384 (in French).
- Livio, M. 2006 *The Equation That Couldn't Be Solved: How Mathematical Genius Discovered the Language of Symmetry*, Simon & Schuster, New York, 368 pages.
- Stewart, I. N. 2015 *Galois Theory*, Chapman and Hall / CRC, Boca Raton, 344 pages.

Keywords: Group theory; Galois.

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