The Chain Rule and Gears

by **Marcio Luis Ferreira Nascimento**

When teaching mathematics it is sometimes important to introduce new mathematical concepts by means of illustrative examples, mainly because many students choose careers in the Arts, Health or Humanities, that habitually use not just calculations. We present a simple explanation of the chain rule in calculus, just observing how simple gear machines, such as for bicycles, work. This idea was used by one of the 'inventors' of calculus, mainly to elaborate one of the first calculator machines.

Machines are formidable objects. No wonder, the German polymath Gottfried Wilhelm Leibniz (1646–1716, see Figure 1) had as one of his favourite pastimes spending hours inventing and designing mechanical machines.

Fig. 1 Gottfried Wilhelm Leibniz (1646–1716), German polymath, from a paint of the German artist Andreas Scheits (1655–1735) in 1703.
Source: Hannover Public Library (*Stadtbibliothek Hannover*: **www.stadtbibliothek-hannover.de**), in public domain

He heard about the French philosopher, inventor, physicist and mathematician, Blaise Pascal (1623–1662), who had succeeded in building a mechanical calculator, named after him as *la pascaline*. Pascal built such an apparatus at the age of 19, and called it his *arithmetic machine*, to assist his father, a French tax supervisor, who had to perform a large number of calculations to complete his work. This device was patented in 1649 as a royal privilege signed by King Louis XIV (1638–1715): *Privilège du Roi, pour la Machine Arithmétique*.

While *la pascaline* only made additions and subtractions, Leibniz proposed, with little information at his disposal, to expand and construct a much better one, including multiplication and division, called *Staffelwalze* (meaning 'stepped drum'). Such a project, which was to proceed the four operations as *leicht*, *geschwind*, *gewiß* ('easy, fast and reliable'), began in 1672 and was completed 22 years later. There is a beautiful replica in *The Deutsches Museum* (**www.deutsches-museum.de**) in Munich.

The principle of any mechanical machine involves gears. The first machines were invented in China and in Greece in ancient times. Perhaps the greatest exponent was Archimedes of Syracuse (287–212 BCE). To transmit a rotation from one cogwheel to another using an intermediate cogwheel (assuming that these are associated with axes), there is a simple rule involving the number of teeth.

Let's begin with the simplest situation in which each gear has the same number of teeth of the same size. They spin with the same speed or rotation. If there are just two gears, they spin in different directions, one clockwise and the other anticlockwise. With three gears in series, the middle one rotates in the opposite direction to the others.

It is possible to invent machines with a huge number of cogwheels or gears. For example, if two coupled gears (or a gear train) have a different number of teeth, that with the largest number (N) will rotate slower (v), while the smaller one (n) will rotate faster (V). Put another way, the product between the number of teeth and speed of each gear is constant: $Nv = nV$, or: $N/n = V/v$. Thus, a larger gear with 12 teeth (named crown), coupled to another with 6 teeth (named pinion), should rotate only at half the speed of the smaller. Mathematically, it is written as follows: $N/n = 12/6 = V/v = 2$. Bicycles follow this simple principle, although one difference is that the two gears are separated – but still coupled by a chain (see Figure 2).

In this way, when one rider pedals, he/she drives a larger toothed crown (about 20 to 30 teeth), which rotates at the distance of a pinion with fewer teeth (and whose axis is that of the wheel itself). When pedalling, one makes a certain number of turns (or spin) that is much smaller than that made by the wheel.

There is a curious rule when using three coupled gears, which Leibniz seems to have been noticed: the rotation of one of two pieces separated by a third depends precisely on the rotation of the centrepiece (or the turning of any of the other pieces, because the system is coupled). As

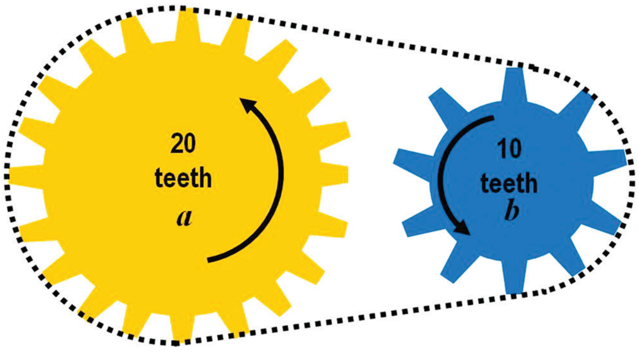

Fig. 2 Schematic device of two coupled gears by a chain (dashed line) as in a bicycle, with a = 20 teeth and b = 10 teeth. The gear spin coupled to the wheel corresponds to twice the spin of those who pedal; or put another way: while someone pedals with a full turn, the wheel should make two turns

an example, if a gear has 12 teeth (c), directly associated with a second one with 6 teeth (b), the third will need 18 teeth (a) for an appropriated system to rotate properly, as follows:

$$\frac{a}{c}=\frac{a}{b}\cdot\frac{b}{c},$$

according to Figure 3.

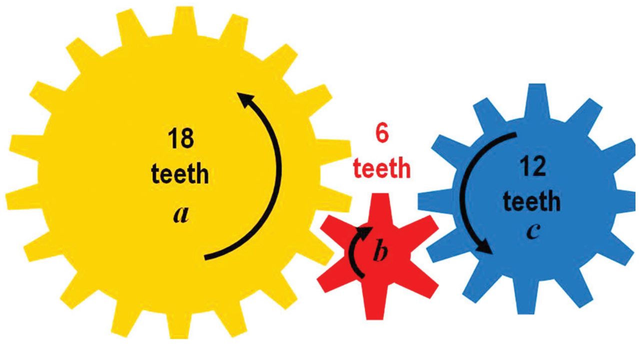

Fig. 3 Schematic device of three coupled gears, with a = 18 teeth, b = 6 teeth and c = 12 teeth

This really happens, because when we replace the a, b and c values, one finds the following:

$$\frac{18}{12}=\frac{18}{6}\cdot\frac{6}{12}.$$

Such ratio notation (or division between two numbers, a/b) was proposed by Leibniz himself in an October, 1674 manuscript, *Schediasma de Methodo Tangentium Inversa ad Circulum Applicata* or "Capricious Inverse Method of Tangents Applied to the Circle" (Child, 2007). In making the above divisions, one obtains the following equation:

$1.5 = 3 \times 0.5$

The relation above reflects the gear train composition: one spin depends on the other, following the product between two numbers relating teeth ratios of different gears.

This result looks like child's play, and these were the exact words of the young thinker and philosopher Leibniz who, until he was thirty, considered himself a layman in mathematics. He had glimpsed his future in a schoolboy essay in 1666, at the age of twenty, after his first doctorate (Bell, 2014). Sometime later, with persistence and some insight, he quickly mastered the basic concepts and was able to propose the fundamentals of what is now named differential and integral calculus.

In the expression

$$\frac{\Delta a}{\Delta c}=\frac{\Delta a}{\Delta b}\cdot\frac{\Delta b}{\Delta c},$$

Δ represents a small variation (increment or decrement, or even step of the toothed part), meaning that if the larger gear, with 18 teeth, does not make a complete turn (i.e. turns only by Δa), there will be a corresponding partial rotation of the last gear ((i.e. Δc). It is possible to rewrite the previous sentence in a rather subtle but equally powerful way:

$$\frac{da}{dc}=\frac{da}{db}\cdot\frac{db}{dc}.$$

This particular ratio notation or division dx/dy was introduced by Leibniz around 1675 (Child, 2007; Bell, 2014), according to manuscripts preserved at the Leibniz Library (**www.gwlb.de**). This last notation was known by Leibniz, where d means a small difference (from Latin, *differentia*), much less than D. Such a value is called a *differential*, and may also be increased or decreased. Leibniz published the dx notation in his first paper on calculus, *Nova Methodus pro Maximis et Minimis, Itemque Tangentibus, quae nec Fractas nec Irrationales Quantitates Moratur, et Singulare pro Illis Calculi Genus* ("New Method for Maxima and Minima, and for Tangents, that is not Hindered by Fractional or Irrational Quantities, and a Singular Kind of Calculus for the Above Mentioned"). It appeared in the first German scientific journal, *Acta Eruditorum* (Leibniz, 1684) of which Leibniz was also the first editor.

The well-known chain rule appeared implicitly in the first calculus book, published anonymously by the French mathematician Guillaume François Antoine de L'Hôpital (1661–1704, see Figure 4), in the masterpiece *Analyse des Infiniment Petits pour la Intelligence des Lignes Courbes* ("Infinitesimal Analysis to Understand Curves") (L'Hospital, 1696).

Indeed, such a book was based on the class notes of a course given to L'Hôpital (Bradley *et al*., 2015) by Leibniz's great friend and former student, Johann Bernoulli (1667–1748) (Rodriguez and Lopez Fernandez, 2010). The modern definition of the chain rule was only published a century later by Joseph Louis Lagrange (1736–1813),

Fig. 4 Guillaume de L'Hôpital (1661–1704), engraving by Gerard Edelinck. [Figure in public domain]

as described in his *Théorie des Fonctions Analytiques* ("Theory of Analytical Functions") (Lagrange, 1797).

It is therefore possible to perceive, not only numerically but also conceptually, that the rotation of coupled gears follows a composition that has a sound mathematical basis, named today as the chain rule, where its parts are attached to a fairly precise Leibnizian relation. The rotation of one gear is a function of the others! And the intersection between two gears corresponds mechanically to what is defined mathematically as the tangent or derivative. It can be concluded therefore that the chain rule presents an ingenious gear.

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